Journal of Statistical Physics, Vol. 79, Nos. 3/4, 1995

Remarks on the Star–Triangle Relation in the Baxter–Bazhanov Model

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Received October 20, 1994

We show that the restricted star–triangle relation introduced by Bazhanov and Baxter can be obtained either from the star–triangle relation of the chiral Potts model or from the star–square relation proposed by Kashaev, Mangazeev, and Stroganov, and give a response to a guess of Bazhanov and Baxter.

KEY WORDS: Three-dimensional integrable lattice models; Baxter–Bazhanov model; restricted star–triangle relations; chiral Potts model; star–square relation.

1. INTRODUCTION

Recently much progress has been made in three-dimensional integrable lattice models. Bazhanov and Baxter generalized the trigonometric Zamolodchikov model with two states[1] to the case of arbitrary N states.[2,3] The star–star relation and the star–square relation of this model are discussed in detail in refs. 3–5. Mangazeev *et al.*[6–8] enlarge the integrable lattice model in three dimensions to the case where the weight functions are parametrized in terms of elliptic functions. Just as the Yang–Baxter equations or the star–triangle relations play a central role in the theory of two-dimensional integrable models, the tetrahedron relations replace the Yang–Baxter equations as the commutativity conditions[9] for the three-dimensional lattice models. The restricted star–triangle relations of the cubic lattice model introduced by Bazhanov and Baxter have the following form:

$$\sum_{l=0}^{N-1} \frac{w(v_2, a-l)}{w(v_1, -l)\,\gamma(b, l)} = \varphi_1(v_1, v_2)\,\frac{w(v_2', -b)\,w(v_2/(\omega v_1), a)}{w(v_1', a-b)} \qquad (1)$$

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$$\sum_{l=0}^{N-1} \frac{w(v_3, -l)\,\gamma(b, l)}{w(v_4, a-l)} = \varphi_2(v_3, v_4)\,\frac{w(v_3', a-b)}{w(v_4', -b)\,w(v_4/v_3, a)} \tag{2}$$

where φ_1 and φ_2 are scalar functions and

$$\frac{w(v, a)}{w(v, 0)} = [\Delta(v)]^a \prod_{j=1}^{a} (1-\omega^j v)^{-1}, \qquad v^N + \Delta^N = 1 \tag{3}$$

$$\omega = \exp(2\pi i/N), \qquad \omega^{1/2} = \exp(i\pi/N), \qquad \gamma(a, b) = \omega^{ab} \tag{4}$$

v_i and v_i' $(i = 1, 2, 3, 4)$ satisfy

$$\begin{aligned} v_1' &= \frac{v_2 \Delta(v_1)}{\omega v_1 \Delta(v_2)}, & \Delta(v_1') &= \frac{\Delta(v_2/(\omega v_1))}{\Delta(v_2)}, \\ v_2' &= \frac{\Delta(v_1)}{\Delta(v_2)}, & \Delta(v_2') &= \frac{\omega v_1 \Delta(v_2/(\omega v_1))}{\Delta(v_2)} \end{aligned} \tag{5}$$

$$\begin{aligned} v_3' &= \frac{v_4 \Delta(v_3)}{v_3 \Delta(v_4)}, & \Delta(v_3') &= \frac{\Delta(v_4/v_3)}{\Delta(v_4)}, \\ v_4' &= \frac{\Delta(v_3)}{\omega \Delta(v_4)}, & \Delta(v_4') &= \frac{v_3 \Delta(v_4/v_3)}{\Delta(v_4)} \end{aligned} \tag{6}$$

Equations (1) and (2) can be changed into each other. Bazhanov and Baxter point out that it is quite possible that Eq. (1) is a particular case of a more general relation and γ is just a limiting value of a more complex function. The purpose of this note is to give a response to this suggestion. In Section 2 the star–triangle relation of the Baxter–Bazhanov model is obtained either from the star–triangle relation of the chiral Potts model or from the star–square relation introduced by Kashaev *et al.* In Section 3, the result is changed into the form of Eqs. (1) and (2). Note that the last relation in Eqs. (5) is different from the original one. The details will be given also in Section 3.

2. THE STAR–TRIANGLE RELATION OF BAXTER–BAZHANOV MODEL

As is well known, the star–triangle relation of the chiral Potts model can be formulated as

$$\sum_{l=1}^{N} \bar{w}_{qr}^{\mathrm{CP}}(m-l)\, w_{pr}^{\mathrm{CP}}(n-l)\, \bar{w}_{pq}^{\mathrm{CP}}(l-k) = R_{pqr} w_{pq}^{\mathrm{CP}}(n-m)\, \bar{w}_{pr}^{\mathrm{CP}}(m-k)\, w_{qr}^{\mathrm{CP}}(n-k) \tag{7}$$

where

$$\begin{aligned}\frac{w_{pq}^{\mathrm{CP}}(n)}{w_{pq}^{\mathrm{CP}}(0)}&=\prod_{j=1}^{n}\frac{d_pb_q-a_pc_q\omega^j}{b_pd_q-c_pa_q\omega^j}\\ \frac{\bar{w}_{pq}^{\mathrm{CP}}(n)}{\bar{w}_{pq}^{\mathrm{CP}}(0)}&=\prod_{j=1}^{n}\frac{\omega a_pd_q-d_pa_q\omega^j}{c_pb_q-b_pc_q\omega^j}\end{aligned} \tag{8}$$

and

$$a_p^N+k'b_p^N=kd_p^N,\qquad k'a_p^N+b_p^N=kc_p^N,\qquad k^2+k'^2=1 \tag{9}$$

Let

$$w(x,y,z\,|\,l)=\prod_{j=1}^{l}\frac{y}{z-x\omega^j},\qquad x^N+y^N=z^N \tag{10}$$

$$w_{pq}(n)\equiv w(\omega^{-1}c_pb_q,d_pa_q,b_pc_q\,|\,n) \tag{11}$$

and define the map R as

$$R:\quad (a_p,b_p,c_p,d_p)\to(b_p,\omega a_p,d_p,c_p) \tag{12}$$

When we set $a_p=d_r=0$, we obtain the following relations:

$$\frac{w_{pr}^{\mathrm{CP}}(n)}{w_{pr}^{\mathrm{CP}}(0)}=w_{pR(r)}(n),\qquad \frac{\bar{w}_{pr}^{\mathrm{CP}}(n)}{\bar{w}_{pr}^{\mathrm{CP}}(0)}=\frac{1}{w_{pr}(-n)} \tag{13}$$

$$\frac{w_{pq}^{\mathrm{CP}}(n)}{w_{pq}^{\mathrm{CP}}(0)}=w_{pR(q)}(n),\qquad \frac{\bar{w}_{pq}^{\mathrm{CP}}(n)}{\bar{w}_{pq}^{\mathrm{CP}}(0)}=\frac{1}{w_{pq}(-n)} \tag{14}$$

$$\frac{w_{qr}^{\mathrm{CP}}(n)}{w_{qr}^{\mathrm{CP}}(0)}=w_{R^{-1}(q)r}(-n),\qquad \frac{\bar{w}_{qr}^{\mathrm{CP}}(n)}{\bar{w}_{qr}^{\mathrm{CP}}(0)}=\frac{1}{w_{qr}(-n)} \tag{15}$$

By taking account of the star–triangle equation (7) of the chiral Potts model, we get

$$\sum_{l=1}^{N}\frac{w_{pR(r)}(n-l)}{w_{qr}(l-m)\,w_{pq}(k-l)}=R'_{pqr}\,\frac{w_{pR(q)}(n-m)\,w_{R^{-1}(q)r}(k-n)}{w_{pr}(k-m)} \tag{16}$$

with $a_p=d_r=0$, where R'_{pqr} is a scalar function. This is just the star-triangle equation of the Baxter–Bazhanov model. If we set $a_p=c_r=0$, similarly we have

$$\sum_{l=1}^{N}\frac{w_{pR(r)}(n+l)}{w_{R(q)\,R(r)}(m+l)\,w_{pq}(k+l)}=\bar{R}'_{pqr}\,\frac{w_{pR(q)}(n-m)\,w_{qR(r)}(n-k)}{w_{R(p)\,R(r)}(m-k)} \tag{17}$$

where $\bar{R}'_{pqr}$ is also a scalar function. Both of the above two equations can be changed into the form of Eqs. (1) and (2). This will be discussed in Section 3. Now we give the connection between Eq. (16) and the star–square relation in the Baxter–Bazhanov model. Let

$$w(x, y, z \mid l) = (y/z)^l\, w(x/z \mid l), \qquad \Phi(a-b) = \omega^{(a-b)(N+a-b)/2} \tag{18}$$

In the version of Kashaev *et al.*[4] the star–square relation can be written as

$$\left\{ \sum_{\sigma \in Z_N} \frac{w(x_1, y_1, z_1 \mid a+\sigma)\, w(x_2, y_2, z_2 \mid b+\sigma)}{w(x_3, y_3, z_3 \mid c+\sigma)\, w(x_4, y_4, z_4 \mid d+\sigma)} \right\}_0$$
$$= \frac{(x_2 y_1/x_1 z_2)^a\, (x_1 y_2/x_2 z_1)^b\, (z_3/y_3)^c\, (z_4/y_4)^d}{\Phi(a-b)\, \omega^{(a+b)/2}}$$
$$\times \frac{w(\omega x_3 x_4 z_1 z_2/x_1 x_2 z_3 z_4 \mid c+d-a-b)}{w\left(\frac{x_4 z_1}{x_1 z_4} \,\middle|\, d-a\right) w\left(\frac{x_3 z_2}{x_2 z_3} \,\middle|\, c-b\right) w\left(\frac{x_3 z_1}{x_1 z_3} \,\middle|\, c-a\right) w\left(\frac{x_4 z_2}{x_2 z_4} \,\middle|\, d-b\right)} \tag{19}$$

where the subscript 0 after the curly brackets indicates that the l.h.s. of the above equation is normalized to unity at zero exterior spins, and the constraint condition $y_1 y_2 z_3 z_4/(z_1 z_2 y_3 y_4) = \omega$ should be imposed owing to spin $\sigma \in Z_N$, but the r.h.s. of the above equation is independent of σ. Set

$$\begin{aligned} &x_1 = c_q b_r, && y_1 = \omega d_q a_r, && z_1 = b_q c_r \\ &x_2 = c_p a_q, && y_2 = d_p b_q, && z_2 = b_p d_q \\ &x_3 = \omega^{-1} c_p b_r, && y_3 = d_p a_r, && z_3 = b_p c_r \\ &x_4 = 0, && y_4 = z_4 && \end{aligned} \tag{20}$$

By considering the "inversion" relation[4,5]

$$\sum_{k \in Z_N} \frac{w(x, y, z \mid k, l)}{w(x, y, \omega z \mid k, m)} = N \delta_{l,m} \frac{1 - z/x}{1 - z^N/x^N} \tag{21}$$

where $\delta_{l,m}$ is the Kronecker symbol on Z_N, we get Eq. (16) from the start–square relation (19). Equation (17) can be obtained similarly.

3. DISCUSSION

First, Eq. (16) can be changed into the form of Eq. (1) and Eq. (2) by using the notations

$$
\begin{gathered}
v_1=\frac{c_p b_q}{\omega b_p c_q},\quad v_2=\frac{b_q c_r}{c_q b_r},\quad \Delta(v_1)=\frac{d_p a_q}{b_p c_q},\quad \Delta(v_2)=\frac{\omega^{1/2} d_q a_r}{c_q b_r} \\
v_1'=\frac{a_q c_r}{d_q b_r},\quad v_2'=\frac{c_p a_q}{b_p d_q},\quad \Delta(v_1')=\frac{\omega^{1/2} c_q a_r}{d_q b_r},\quad \Delta(v_2')=\frac{d_p b_q}{b_p d_q} \\
\Delta\left(\frac{v_2}{\omega v_1}\right)=\frac{\omega^{1/2} d_p a_r}{c_p b_r}
\end{gathered}
\tag{22}
$$

and

$$
\begin{gathered}
v_3=\frac{b_q c_r}{c_q b_r},\quad v_4=\frac{c_p b_q}{\omega b_p c_q},\quad \Delta(v_3)=\frac{\omega^{1/2} d_q a_r}{c_q b_r},\quad \Delta(v_4)=\frac{d_p a_q}{b_p c_q} \\
v_3'=\frac{d_q b_r}{\omega a_q c_r},\quad v_4'=\frac{b_p d_q}{\omega c_p a_q},\quad \Delta(v_3')=\frac{c_q a_r}{a_q c_r},\quad \Delta(v_4')=\frac{d_p b_q}{\omega^{1/2} c_p a_q}
\end{gathered}
\tag{23}
$$

respectively, with $d_p b_r=\omega^{1/2} c_p a_r$. The v_i and v_i' $(i=1,2,3,4)$ satisfy Eqs. (5) and (6). Here we show that the last relation in Eq. (5) is correct and this relation is different from the one in ref. 3. Equation (17) also can be changed into Eq. (1) by setting

$$
\begin{gathered}
v_1=\frac{b_p d_q}{\omega c_p a_q},\qquad v_2=\frac{d_q a_r}{a_q d_r}, \\
\Delta(v_1)=\frac{d_p b_q}{\omega^{1/2} c_p a_q},\qquad \Delta(v_2)=\frac{c_q b_r}{a_q d_r} \\
v_1'=\frac{c_p b_q}{\omega b_p c_q},\qquad v_2'=\frac{b_q d_r}{\omega c_q a_r}, \\
\Delta(v_1')=\frac{d_p a_q}{b_p c_q},\qquad \Delta(v_2')=\frac{d_q b_r}{\omega^{1/2} c_q a_r} \\
\Delta\left(\frac{v_2}{\omega v_1}\right)=\frac{d_p b_r}{b_p d_r}
\end{gathered}
\tag{24}
$$

with $c_p b_r=\omega^{1/2} d_p a_r$. Similarly, Eq. (2) can be obtained easily from Eq. (17). In fact, each of the relations (16) and (17) is a corollary of the other by taking account of the "inversion" relation (21).

In summary, in this note we obtained the star–triangle relation of the Baxter–Bazhanov model from the star–triangle relation of the chiral Potts model and responded to a guess proposed by Bazhanov and Baxter. We also found a connection between the star–triangle relation and the star–square relation in the Baxter–Bazhanov model.

ACKNOWLEDGMENTS

One of the authors (Z.-N. Hu) would like to thank H. Y. Guo, P. Wang, and K. Wu for interesting discussions.

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Communicated by P. J. Forrester